

Model order reduction and sparse orthogonal expansions for random linear dynamical systems

Roland Pulch

Department of Mathematics and Computer Science,
Ernst-Moritz-Arndt-Universität Greifswald,
Walther-Rathenau-Str. 47, 17489 Greifswald, Germany.
Email: pulchr@uni-greifswald.de

Abstract

We consider linear dynamical systems of ordinary differential equations or differential algebraic equations. Physical parameters are substituted by random variables for an uncertainty quantification. We expand the state variables as well as a quantity of interest into an orthogonal system of basis functions, which depend on the random variables. For example, polynomial chaos expansions are applicable. The stochastic Galerkin method yields a larger linear dynamical system, whose solution approximates the unknown coefficients in the expansions. The Hardy norms of the transfer function provide information about the input-output behaviour of the Galerkin system. We investigate two approaches to construct a sparse representation of the quantity of interest, where just a low number of coefficients is non-zero. Firstly, a standard basis is reduced by the omission of basis functions, whose accompanying Hardy norms are relatively small. Secondly, a projection-based model order reduction is applied to the Galerkin system and allows for the definition of new basis functions as a sparse representation. In both cases, we prove error bounds on the sparse approximation with respect to Hardy norms. Numerical experiments are demonstrated for a test example modelling a linear electric circuit.

Key words: linear dynamical systems, orthogonal expansion, polynomial chaos, model order reduction, transfer function, Hardy norms.

1 Introduction

In science and engineering, mathematical modelling often yields dynamical systems of ordinary differential equations (ODEs) or differential algebraic equations (DAEs). We focus on linear time-invariant dynamical systems. A quantity of interest is defined as an output of the problem. Physical parameters of the systems may exhibit uncertainties due to measurement errors or imperfections of an industrial manufacturing, for example. The uncertainties are described by the introduction of random variables. Since many parameters often appear in a system, we are interested in the case of high numbers of random variables.

We expand the state variables as well as the quantity of interest into an orthonormal system of basis functions depending on the random variables. For example, the expansions of the polynomial chaos can be used, see [2, 9, 27, 29]. Our aim is the construction of a sparse approximation to the random quantity of interest, where only a few basis functions are required for a sufficiently accurate representation. Equivalently, we identify a low-dimensional subspace, which allows for a good approximation. Several previous works exist concerning the computation of such a sparse approximation. As a tool was used, for example, least angle regression [5], sparse grid quadrature [6], compressed sensing [7] and ℓ_1 -minimisation [15, 16]. Our task can also be seen as an identification of a stochastic reduced basis, which was examined for random linear systems of algebraic equations in [18, 24]. On the one hand, some methods start from a small set of basis functions and extend the basis successively until the approximation becomes sufficiently accurate. On the other hand, some techniques require the choice of an initial set of basis functions, which is large and often overprecise, and reduce this basis. We apply strategies of the latter type.

Either a stochastic Galerkin method or a stochastic collocation technique yields approximations to the unknown coefficient functions in the expansions, see [28, 29]. In this paper, we employ the stochastic Galerkin approach, which induces a larger linear dynamical system with many outputs. Hardy norms provide a measure for the importance of each output, where the \mathcal{H}_2 -norm and \mathcal{H}_∞ -norm are used. Since the system becomes huge for large numbers of random parameters, a high potential for a model order reduction (MOR) appears. General theory on MOR can be found in the monographs [1, 3, 25], for example. We focus on projection-based techniques for the reduction of linear dynamical systems, see [10, 11, 13, 14]. Projection-based MOR was applied to the stochastic Galerkin system in the previous works [17, 21, 22, 23, 30].

We investigate two strategies for the construction of a sparse approximation. Firstly, a large initial basis is reduced by neglecting outputs of the Galerkin system with relatively low Hardy norms. This reduction implies directly a sparse

approximation to the random quantity of interest. Secondly, a general projection-based MOR technique decreases the dimensionality of the Galerkin system. We show that this MOR allows for the identification of a sparse approximation to the random quantity of interest provided that the reduction achieves a sufficiently small system. In both cases, error bounds are proved for the sparse approximations with respect to Hardy norms.

The paper is organised as follows. In Section 2, we introduce the problem formulation and review already existing theory to be applied. The construction of a sparse approximation by omitting basis functions is examined in Section 3. The definition of new basis functions using the information from an MOR is discussed in Section 4. We present numerical results for an illustrative example in Section 5.

2 Problem definition

In this section, we define the problem under investigation. Furthermore, results from previous literature, which are relevant for our approaches, are outlined.

2.1 Linear dynamical systems

We consider a linear time-invariant system in descriptor form

$$\begin{aligned} E(p)\dot{x}(t,p) &= A(p)x(t,p) + B(p)u(t) \\ y(t,p) &= C(p)x(t,p), \end{aligned} \tag{1}$$

where the matrices $A, E \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_{\text{in}}}$ and $C \in \mathbb{R}^{n_{\text{out}} \times n}$ depend on physical parameters $p \in \Pi \subseteq \mathbb{R}^q$. The input $u : [0, \infty) \rightarrow \mathbb{R}^{n_{\text{in}}}$ is supplied, while the output is defined by $y : [0, \infty) \times \Pi \rightarrow \mathbb{R}^{n_{\text{out}}}$. If the matrix E is regular, then the system (1) consists of ODEs with state variables $x : [0, \infty) \times \Pi \rightarrow \mathbb{R}^n$. In our analysis, initial values $x(0, p) = 0$ are supposed for all $p \in \Pi$. If the matrix E is singular, then the system (1) represents DAEs with inner variables x . We restrict ourselves to the case of single-input-single-output (SISO) with $n_{\text{in}} = n_{\text{out}} = 1$, because generalisations to multiple-input-multiple-output (MIMO) are straightforward. We assume that the matrix pencil $\lambda E(p) - A(p)$ is regular for each $p \in \Pi$. Moreover, let the system (1) be stable for all $p \in \Pi$, i.e., the finite eigenvalues $\Sigma(p) \subset \mathbb{C}$ of the matrix pencil $\lambda E(p) - A(p)$ exhibit a negative real part.

The transfer function $H : (\mathbb{C} \setminus \Sigma(p)) \rightarrow \mathbb{C}$ characterises the input-output behaviour of the SISO system (1) in the frequency domain, see [1, Eq. (4.22)] for

explicit ODEs or [11, Eq. (2.8)] for DAEs. This transfer function reads as

$$H(s, p) := C(p) (sE(p) - A(p))^{-1} B(p) \quad \text{for } s \in \mathbb{C} \setminus \Sigma(p). \quad (2)$$

In the case of an SISO system (1), the transfer function becomes a rational function with respect to the frequency s .

2.2 Stochastic modelling and orthogonal expansions

We substitute the parameters $p \in \Pi$ of the system (1) by independent random variables $p : \Omega \rightarrow \Pi$ on some probability space $(\Omega, \mathcal{A}, \mu)$ with event space Ω , sigma-algebra \mathcal{A} and probability measure μ . Let a joint probability density function $\rho : \Pi \rightarrow \mathbb{R}$ be given. For a measurable function $f : \Pi \rightarrow \mathbb{R}$, the expected value reads as

$$\mathbb{E}[f] := \int_{\Omega} f(p(\omega)) \, d\mu(\omega) = \int_{\Pi} f(p) \rho(p) \, dp$$

provided that the integral is finite. The Hilbert space

$$\mathcal{L}^2(\Pi, \rho) := \{f : \Pi \rightarrow \mathbb{R} : f \text{ measurable and } \mathbb{E}[f^2] < \infty\}$$

is equipped with the inner product

$$\langle f, g \rangle := \mathbb{E}[fg] := \int_{\Pi} f(p)g(p)\rho(p) \, dp \quad \text{for } f, g \in \mathcal{L}^2(\Pi, \rho).$$

We write the associated norm as

$$\|f\|_{\mathcal{L}^2(\Omega)} := \sqrt{\langle f, f \rangle}$$

with $\mathcal{L}^2(\Omega)$ as an abbreviation for $\mathcal{L}^2(\Pi, \rho)$. Concerning the dynamical system (1), we assume that $x_1(t, \cdot), \dots, x_n(t, \cdot), y(t, \cdot) \in \mathcal{L}^2(\Pi, \rho)$ pointwise for $t \in [0, \infty)$.

Now let a complete orthonormal system $(\Phi_i)_{i \in \mathbb{N}} \subset \mathcal{L}^2(\Pi, \rho)$ be given. It holds that $\langle \Phi_i, \Phi_j \rangle = 0$ for $i \neq j$ and $\langle \Phi_i, \Phi_j \rangle = 1$ for $i = j$. We assume that the first basis function is always the constant $\Phi_1 \equiv 1$. In the polynomial chaos (PC), orthogonal polynomials are applied, see [9, 27, 29]. The orthonormal multivariate polynomials are just the products of the univariate orthonormal polynomials for each random variable. It follows that the expansions

$$x(t, p) = \sum_{i=1}^{\infty} v_i(t) \Phi_i(p) \quad \text{and} \quad y(t, p) = \sum_{i=1}^{\infty} w_i(t) \Phi_i(p), \quad (3)$$

where the coefficient functions $v_i : [0, \infty) \rightarrow \mathbb{R}^n$ and $w_i : [0, \infty) \rightarrow \mathbb{R}$ are defined by

$$v_{i,j}(t) = \langle x_j(t, \cdot), \Phi_i(\cdot) \rangle \quad \text{and} \quad w_i(t) = \langle y(t, \cdot), \Phi_i(\cdot) \rangle, \quad (4)$$

converge in $\mathcal{L}^2(\Pi, \rho)$ pointwise for $t \in [0, \infty)$ and $j = 1, \dots, n$.

2.3 Sparse orthogonal representations

Concerning the series (3), finite approximations result by a truncation. We employ the approximations

$$x^{(\mathcal{I})}(t, p) = \sum_{i \in \mathcal{I}} v_i(t) \Phi_i(p) \quad \text{and} \quad y^{(\mathcal{I})}(t, p) = \sum_{i \in \mathcal{I}} w_i(t) \Phi_i(p) \quad (5)$$

with a finite set of indices $\mathcal{I} \subseteq \mathbb{N}$. We focus on the output of the random dynamical system (1) as quantity of interest. The associated truncation error reads as

$$\|y(t, \cdot) - y^{(\mathcal{I})}(t, \cdot)\|_{\mathcal{L}^2(\Omega)} = \sqrt{\sum_{i \notin \mathcal{I}} w_i(t)^2} \quad (6)$$

for each $t \geq 0$ due to the orthonormality of the basis functions.

We consider some set \mathcal{I} , which is assumed to generate a highly accurate approximation to the output. If multivariate polynomials are applied in the expansion, then often all polynomials up to a total degree d are included, which yields the index set, cf. [5, Sect. 2],

$$\mathcal{I}_d := \left\{ i : \Phi_i(p) = \phi_{j_1}^{(1)}(p_1) \cdots \phi_{j_q}^{(q)}(p_q) \text{ with } j_1 + \cdots + j_q \leq d \right\}. \quad (7)$$

Therein, the systems $(\phi_j^{(\ell)})_{j \in \mathbb{N}_0}$ include the univariate orthonormal polynomials with respect to the ℓ th random variable and the degree of $\phi_j^{(\ell)}$ is exactly j . The number of basis polynomials becomes, see [29, Eq. (5.24)],

$$|\mathcal{I}_d| = \frac{(q+d)!}{q!d!}.$$

Thus the number of basis functions is large for high numbers q of random variables even if the total degree is low, say $d \leq 5$.

We start from a (usually large) initial set \mathcal{I} in (5), where the truncation error (6) is below some given threshold $\delta > 0$. For polynomial expansions, the set \mathcal{I}_d with the smallest total degree d satisfying this accuracy can be chosen. Our aim is to identify a sparse approximation by one of the following two strategies.

1. For high dimensions q , often an index set $\mathcal{I}' \subset \mathcal{I}$ exists with $|\mathcal{I}'| \ll |\mathcal{I}|$, while the truncation error (6) is still lower than δ for $y^{(\mathcal{I}')}$. Since the new approximation can be written in the form

$$y^{(\mathcal{I}')} (t, p) = \sum_{i \in \mathcal{I}} \tilde{w}_i(t) \Phi_i(p) \quad \text{with} \quad \tilde{w}_i = \begin{cases} w_i & \text{for } i \in \mathcal{I}', \\ 0 & \text{for } i \notin \mathcal{I}', \end{cases}$$

where most of the coefficients are equal to zero, $y^{(\mathcal{I}')}$ is called a sparse representation. A measure for the sparsity is the ratio $\sigma \in (0, 1)$ defined by $\sigma := |\mathcal{I}'|/|\mathcal{I}|$, see [5, Eq. (30)].

2. A new finite-dimensional subspace spanned by the multivariate orthonormal polynomials $\{\Psi_1, \dots, \Psi_r\}$ is constructed, i.e.,

$$\text{span}\{\Psi_1, \dots, \Psi_r\} \subset \text{span}\{\Phi_i : i \in \mathcal{I}\}.$$

The alternative approximation reads as

$$y^{(r)}(t, p) = \sum_{i=1}^r \tilde{w}_i(t) \Psi_i(p)$$

with its own coefficient functions $\tilde{w}_1, \dots, \tilde{w}_r$ to be defined. The aim is to keep the dimension r as small as possible such that the $\mathcal{L}^2(\Omega)$ -norm of the error for this approximation is still lower than δ . Again the sparsity is measured by $\sigma := r/|\mathcal{I}|$, if the initial set \mathcal{I} is used to construct the approximation.

Since the first strategy can be seen as a special case of the second strategy, where the functions $\{\Psi_1, \dots, \Psi_r\}$ represent just a subset of $\{\Phi_i : i \in \mathcal{I}\}$, we expect a higher potential for a dimension reduction of the subspaces in the second approach.

2.4 Stochastic Galerkin method

The unknown coefficient functions in (5) can be determined approximately either by a stochastic collocation technique or a stochastic Galerkin method, see [28, 29]. In this paper, we investigate the stochastic Galerkin method. The approach yields a linear dynamical system

$$\begin{aligned} \hat{E} \dot{\hat{v}}(t) &= \hat{A} \hat{v}(t) + \hat{B} u(t) \\ \hat{w}(t) &= \hat{C} \hat{v}(t) \end{aligned} \tag{8}$$

for $t \geq 0$ of the larger dimension mn for the inner variables, where $m := |\mathcal{I}|$ denotes the cardinality of the index set. Therein, $\hat{v} = (\hat{v}_{i_1}, \dots, \hat{v}_{i_m}) \in \mathbb{R}^{mn}$ and $\hat{w} = (\hat{w}_{i_1}, \dots, \hat{w}_{i_m}) \in \mathbb{R}^m$ represent approximations to the exact coefficient functions (4). The initial values are $\hat{v}(0) = 0$ due to $x(0, p) = 0$ for all p . The constant matrices $\hat{A}, \hat{E} \in \mathbb{R}^{mn \times mn}$, $\hat{B} \in \mathbb{R}^{mn}$ and $\hat{C} \in \mathbb{R}^{m \times mn}$ follow directly from the matrices in the dynamical system (1) and the probability distribution of the parameters by integrals in the random space. For more details on the derivation of the system (8), see [22, Sect. 2.4], for example.

The linear dynamical system (1) is assumed to be SISO, whereas the Galerkin system (8) becomes single-input-multiple-output (SIMO). The system (8) may be unstable even though all original systems (1) are stable, see [26]. However, this loss of stability appears rather seldom. The larger dynamical system (8) exhibits an own input-output behaviour with a transfer function $\hat{H} : (\mathbb{C} \setminus \Theta) \rightarrow \mathbb{C}^n$ for some finite set of poles $\Theta \subset \mathbb{C}$ given by

$$\hat{H}(s) := \hat{C} \left(s\hat{E} - \hat{A} \right)^{-1} \hat{B} \quad \text{for } s \in \mathbb{C} \setminus \Theta. \quad (9)$$

In the vector $\hat{H} = (\hat{H}_1, \dots, \hat{H}_m)^\top$, the components represent approximations to the coefficient functions of an orthogonal expansion for the original transfer function (2) in the frequency domain, see [21, Sect. 3.3].

2.5 Model order reduction

Since the system (8) of the stochastic Galerkin method is huge for high numbers of random parameters and index sets like (7), it represents an excellent candidate for a model order reduction (MOR). A reduction of a stochastic Galerkin system was investigated in [17, 21, 22, 30] using Krylov subspace methods and in [23] using balanced truncation. The task consists in the construction of a smaller dynamical system

$$\begin{aligned} \bar{E}\dot{\bar{v}}(t) &= \bar{A}\bar{v}(t) + \bar{B}u(t) \\ \bar{w}(t) &= \bar{C}\bar{v}(t) \end{aligned} \quad (10)$$

with dimension $r \ll nm$ of the inner variables and the same number of outputs m . Hence the sizes of the matrices are $\bar{A}, \bar{E} \in \mathbb{R}^{r \times r}$, $\bar{B} \in \mathbb{R}^r$, $\bar{C} \in \mathbb{R}^{m \times r}$. In projection-based model reduction, projection matrices $T_l, T_r \in \mathbb{R}^{mn \times r}$ are determined and the reduced matrices read as

$$\bar{A} = T_l^\top \hat{A} T_r, \quad \bar{B} = T_l^\top \hat{B}, \quad \bar{C} = \hat{C} T_r, \quad \bar{E} = T_l^\top \hat{E} T_r. \quad (11)$$

The property $\hat{w}(t) \approx \bar{w}(t)$ for $t \geq 0$ is desired with respect to some norm in the time domain. The reduced system (10) yields

$$\bar{y}^{(I)}(t, p) = \sum_{i \in I} \bar{w}_i(t) \Phi_i(p) \quad (12)$$

as an approximation to the random quantity of interest.

Let $\bar{H} = (\bar{H}_1, \dots, \bar{H}_m)^\top$ be the transfer function of the reduced system (10). The difference between two transfer functions can be quantified by Hardy norms,

see [1, Ch. 5.7]. We apply these norms componentwise in the following. In the time domain, the usual integral norm

$$\|u\|_{\mathcal{L}^2[0,\infty)} = \sqrt{\int_0^\infty u(t)^2 dt}$$

of the Hilbert space $\mathcal{L}^2[0, \infty)$ is used. The next lemma recalls some definitions as well as general results on the input-output relations of linear dynamical systems. The proof follows from the statements in [8, Ch. 2]. Associated error measures are also illustrated in [4, Sect. 2.4].

Lemma 1 *Let the initial values of a linear dynamical system be zero. If G is the transfer function of a stable and proper SISO system, then the \mathcal{H}_∞ -norm*

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} |G(i\omega)| \quad (13)$$

exists and the input-output exhibits the bound

$$\|y\|_{\mathcal{L}^2[0,\infty)} \leq \|G\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}^2[0,\infty)}. \quad (14)$$

If G is the transfer function of a stable and strictly proper SISO system, then the \mathcal{H}_2 -norm

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(i\omega)|^2 d\omega} \quad (15)$$

exists and the input-output satisfies

$$\sup_{t \geq 0} |y(t)| \leq \|G\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}^2[0,\infty)}. \quad (16)$$

We will consider the m outputs of the stochastic Galerkin system (8) separately as SISO systems, which allows for using Lemma 1 in Section 3 and Section 4.

3 Sparsification of a given basis

In this section, we suppose that a large canonical basis is given a priori, where the representation is shortened by neglecting basis functions.

3.1 Selection of a reduced basis

For the quantity of interest, the stochastic Galerkin method yields the approximation

$$\hat{y}^{(\mathcal{I})}(t, p) = \sum_{i \in \mathcal{I}} \hat{w}_i(t) \Phi_i(p). \quad (17)$$

Now we choose a subset $\mathcal{I}' \subset \mathcal{I}$ or, equivalently, a selection of the outputs from the larger coupled system (8). The new approximation reads as

$$\tilde{y}^{(\mathcal{I}')} (t, p) = \sum_{i \in \mathcal{I}'} \hat{w}_i(t) \Phi_i(p). \quad (18)$$

We denote this function by $\tilde{y}^{(\mathcal{I}')}$, because $\hat{y}^{(\mathcal{I})}$ is devoted to the solution from the stochastic Galerkin method with respect to the index set \mathcal{I} . The total error can be estimated by

$$\begin{aligned} \|y(t, \cdot) - \tilde{y}^{(\mathcal{I}')} (t, \cdot)\|_{\mathcal{L}^2(\Omega)} &\leq \|y(t, \cdot) - y^{(\mathcal{I})}(t, \cdot)\|_{\mathcal{L}^2(\Omega)} \\ &\quad + \|y^{(\mathcal{I})}(t, \cdot) - \hat{y}^{(\mathcal{I})}(t, \cdot)\|_{\mathcal{L}^2(\Omega)} \\ &\quad + \|\hat{y}^{(\mathcal{I})}(t, \cdot) - \tilde{y}^{(\mathcal{I}')} (t, \cdot)\|_{\mathcal{L}^2(\Omega)} \end{aligned}$$

for each $t \geq 0$, where the upper bound consists of three terms: the truncation error, the error of the Galerkin method and an additional sparsification error. We assume that the set \mathcal{I} is chosen sufficiently large such that the truncation error as well as the Galerkin error are negligible.

The following theorem highlights the properties of the reduction in this section.

Theorem 1 *The error between the quantity of interest (17) and its sparse approximation (18) exhibits the estimates*

$$\sup_{t \geq 0} \|\hat{y}^{(\mathcal{I})}(t, \cdot) - \tilde{y}^{(\mathcal{I}')} (t, \cdot)\|_{\mathcal{L}^2(\Omega)} \leq \sqrt{\sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \|\hat{H}_i\|_{\mathcal{H}_2}^2} \|u\|_{\mathcal{L}^2[0, \infty)} \quad (19)$$

$$\|\hat{y}^{(\mathcal{I})} - \tilde{y}^{(\mathcal{I}')} \|_{\mathcal{L}^2(\Omega) \times \mathcal{L}^2[0, \infty)} \leq \sqrt{\sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \|\hat{H}_i\|_{\mathcal{H}_\infty}^2} \|u\|_{\mathcal{L}^2[0, \infty)} \quad (20)$$

provided that the Galerkin system (8) is stable as well as strictly proper or proper, respectively.

Proof:

The error of the sparse approximation satisfies the estimate

$$\|\hat{y}^{(\mathcal{I})}(t, \cdot) - \tilde{y}^{(\mathcal{I}')} (t, \cdot)\|_{\mathcal{L}^2(\Omega)} = \sqrt{\sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \hat{w}_i(t)^2} \quad \text{for each } t \geq 0$$

due to the orthonormality of the basis functions.

Firstly, using estimate (16) from Lemma 1, we obtain

$$\sup_{t \geq 0} \left\| \hat{y}^{(\mathcal{I})}(t, \cdot) - \tilde{y}^{(\mathcal{I}')} (t, \cdot) \right\|_{\mathcal{L}^2(\Omega)}^2 \leq \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \sup_{t \geq 0} \hat{w}_i(t)^2 \leq \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \left\| \hat{H}_i \right\|_{\mathcal{H}_2}^2 \|u\|_{\mathcal{L}^2[0, \infty)}^2.$$

Taking the square root yields the statement.

Secondly, the estimate (14) from Lemma 1 implies

$$\begin{aligned} \left\| \hat{y}^{(\mathcal{I})} - \tilde{y}^{(\mathcal{I}')} \right\|_{\mathcal{L}^2(\Omega) \times \mathcal{L}^2[0, \infty)}^2 &= \int_0^\infty \int_{\Pi} \left(\hat{y}^{(\mathcal{I})}(t, p) - \tilde{y}^{(\mathcal{I}')} (t, p) \right)^2 \rho(p) \, dp \, dt \\ &= \int_0^\infty \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \hat{w}_i(t)^2 \, dt = \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \|\hat{w}_i\|_{\mathcal{L}^2[0, \infty)}^2 \leq \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \left\| \hat{H}_i \right\|_{\mathcal{H}_\infty}^2 \|u\|_{\mathcal{L}^2[0, \infty)}^2. \end{aligned}$$

Again the square root operation shows the claimed estimate. \square

The involved Hardy norms can be computed a priori from the matrices in the Galerkin system (8). On the one hand, the best choice of an index set \mathcal{I}' with exactly k elements for minimising the upper error bound (19) or (20) reads as

$$\mathcal{I}' = \arg \min \left\{ \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \left\| \hat{H}_i \right\|^2 : |\mathcal{I}'| = k \right\}.$$

On the other hand, the smallest index set \mathcal{I}' , where the upper error bound (19) or (20) is below a predetermined threshold $\delta > 0$ for a unit norm of the input becomes

$$\mathcal{I}' = \arg \min \left\{ |\mathcal{I}'| : \sum_{i \in \mathcal{I} \setminus \mathcal{I}'} \left\| \hat{H}_i \right\|^2 < \delta^2 \right\}.$$

A drawback of this approximation is that the reduction is based on an upper bound of the sparsification error, which is not sharp in general.

A measure for the quality of the approximation in a relative sense (input has unit norm) is defined as follows. Let $\mathcal{I}'_r \subseteq \mathcal{I}$ be an index set associated to the r largest norms. Although this set is not always unique, the resulting sum of the largest norms is unique. We arrange the ratio

$$\theta_r := \left(\sum_{i \in \mathcal{I}'_r} \left\| \hat{H}_i \right\|^2 \right)^{\frac{1}{2}} / \left(\sum_{i \in \mathcal{I}} \left\| \hat{H}_i \right\|^2 \right)^{\frac{1}{2}} \quad \text{for } r = 1, \dots, |\mathcal{I}|. \quad (21)$$

Obviously, it holds that $0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \dots \leq \theta_{|\mathcal{I}|} = 1$.

3.2 Reduction of the stochastic Galerkin system

The sparse approximation, which is described in Section 3.1, represents a simplification of the representation for the quantity of interest. If still the system (8) from the stochastic Galerkin method is solved, then computational effort is not saved yet. An idea for a workload reduction is to repeat the stochastic Galerkin approach for the smaller index set \mathcal{I}' instead of the initial index set \mathcal{I} . Consequently, a smaller system (8) appears with dimension $|\mathcal{I}'|N$ instead of $|\mathcal{I}|N$ and $|\mathcal{I}'|$ outputs instead of $|\mathcal{I}|$ outputs. If the matrices of the system (8) have already been computed for the set \mathcal{I} , then we obtain the reduced system just by discarding the rows and columns of the matrices, which do not belong to basis functions in the set \mathcal{I}' . However, this approach for a reduction is critical due to two reasons:

1. In the Galerkin approach, the subspace spanned by $\{\Phi_i : i \in \mathcal{I}\}$ is required for both the approximation to the exact solution and to keep the associated residual small. The best approximation to the solution may still be sufficiently accurate in the reduced basis, while the error of the Galerkin method can increase too much.
2. The reduction relies on information about the output only. The inner variables could exhibit a different behaviour with respect to the stochastic modes. Thus crucial interactions of the inner variables may be removed by the downsizing of the system matrices.

Nevertheless, we apply this reduction straightforward and check the reduction error a posteriori by the Hardy norms of the difference in the transfer functions. Removing rows and columns in the matrices of the system (8) yields a system (10) with matrices (11), where the projection matrix $T_1 = T_r$ results from taking rows and columns out of a square identity matrix. Hence we obtain a special case of a projection-based MOR. An additional modification is required in the output matrix, since the number of outputs is assumed to be the identical in the systems (8) and (10). Thus we do not delete rows but change those rows into zero vectors. It follows that

$$\bar{C} = D\hat{C}T_r \quad (22)$$

with a square diagonal matrix D consisting of zeros and ones. Now the quantity of interest $\hat{y}^{(\mathcal{I}')}$ from the smaller Galerkin system (8) is identical to $\bar{y}^{(\mathcal{I})}$ in (12) from the reduced system (10).

We specify bounds on the difference between the random quantity of interest resulting from general systems (8) and (10) in the following theorem. These estimates are also crucial within Section 4.

Theorem 2 *The difference between the output from the stochastic Galerkin system (8) and the output from the reduced system (10) satisfies the estimates*

$$\sup_{t \geq 0} \left\| \hat{y}^{(\mathcal{I})}(t, \cdot) - \bar{y}^{(\mathcal{I})}(t, \cdot) \right\|_{\mathcal{L}^2(\Omega)} \leq \sqrt{\sum_{i \in \mathcal{I}} \left\| \hat{H}_i - \bar{H}_i \right\|_{\mathcal{H}_2}^2} \|u\|_{\mathcal{L}^2[0, \infty)} \quad (23)$$

$$\left\| \hat{y}^{(\mathcal{I})} - \bar{y}^{(\mathcal{I})} \right\|_{\mathcal{L}^2(\Omega) \times \mathcal{L}^2[0, \infty)} \leq \sqrt{\sum_{i \in \mathcal{I}} \left\| \hat{H}_i - \bar{H}_i \right\|_{\mathcal{H}_\infty}^2} \|u\|_{\mathcal{L}^2[0, \infty)} \quad (24)$$

provided that the involved systems are stable and the norms are finite.

The proof of these estimates is straightforward by repeating the concept of the proof for Theorem 1.

Since the smaller Galerkin system yields the quantity of interest $\hat{y}^{(\mathcal{I}')} = \bar{y}^{(\mathcal{I})}$, it holds that $\bar{H}_i = 0$ for $i \notin \mathcal{I}'$ due to the definition of the output matrix (22). Consequently, the upper error estimates (23) and (24) are bounded from below by

$$\sum_{i \in \mathcal{I}} \left\| \hat{H}_i - \bar{H}_i \right\| \geq \sum_{i \notin \mathcal{I}'} \left\| \hat{H}_i \right\|^2. \quad (25)$$

The approximation quality of the smaller Galerkin system is limited by the rate of decay of the Hardy norms in the original Galerkin system.

4 Construction of a small basis by MOR

We discuss the derivation of an alternative basis for a sparse approximation now.

4.1 Definition of basis functions

The system (10) from the MOR should feature a dimensionality $r \ll nm$, where $m = |\mathcal{I}|$ is the number of basis functions included in (5). In the case of a high-dimensional parameter space, the number m becomes huge. Often the MOR techniques are very efficient and thus produce a sufficiently accurate system with a dimensionality $r \ll m$. In this case, we construct an alternative representation for the quantity of interest. Using the random process (12) from the reduced system (10), it holds that

$$\bar{y}^{(\mathcal{I})}(t, p) = \sum_{i \in \mathcal{I}} \bar{w}_i(t) \Phi_i(p) = \sum_{i \in \mathcal{I}} \left[\sum_{j=1}^r \bar{c}_{ij} \bar{v}_j(t) \right] \Phi_i(p) = \sum_{j=1}^r \bar{v}_j(t) \left[\sum_{i \in \mathcal{I}} \bar{c}_{ij} \Phi_i(p) \right].$$

Now we define new functions on the parameter space Π by

$$\Psi_j(p) := \sum_{i \in \mathcal{I}} \bar{c}_{ij} \Phi_i(p) \quad \text{for } j = 1, \dots, r. \quad (26)$$

It is rather unlikely that the functions (26) are linearly dependent. However, the r columns of the matrix \bar{C} often do not exhibit a full numerical rank with respect to a standard machine precision, even if the projection matrix T_r in (11) has full (numerical) rank. This property allows for a small amount of an additional basis reduction to be analysed in the following subsection.

From above, we obtain the sparse representation

$$\bar{y}^{(\mathcal{I})}(t, p) = \sum_{j=1}^r \bar{v}_j(t) \Psi_j(p) \quad (27)$$

with no sparsification error but only the error of the reduction. The sparsity is measured by $\sigma = r/m$. The low-dimensional system (10) yields both this sparse formulation and a cheap simulation in the time domain to obtain the coefficient functions $\bar{v}_1, \dots, \bar{v}_r$. The bounds of Theorem 2 are valid for the difference between the sparse approximation (27) from the system (10) and the quantity of interest (17) from the system (8).

4.2 Transformation to an orthonormal basis

The basis $\{\Psi_1, \dots, \Psi_r\}$ from (26) is not orthogonal. Nevertheless, we can transform this basis to an orthonormal description using the matrix $\bar{C} \in \mathbb{R}^{m \times r}$ of the reduced system (10) only. Let $\Phi := (\Phi_1, \dots, \Phi_m)^\top$ and $\Psi := (\Psi_1, \dots, \Psi_r)^\top$ be column vectors. It holds that $\Psi = \bar{C}^\top \Phi$ due to (26).

We apply a singular value decomposition (SVD), see [12, Ch. 2.5], as a tool for two purposes: to construct an orthonormal basis and to remove unessential parts associated to a numerical rank deficiency of the output matrix. The SVD of the output matrix reads as

$$\bar{C} = USQ \quad \text{with} \quad S = \begin{pmatrix} S' \\ 0 \end{pmatrix} \quad (28)$$

including orthogonal matrices $U \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{r \times r}$ and a matrix $S \in \mathbb{R}^{m \times r}$, whose part $S' \in \mathbb{R}^{r \times r}$ is a diagonal matrix with the singular values $s_1 \geq s_2 \geq \dots \geq s_r > 0$.

It is straightforward to show the equality

$$\Psi^* := S'^{-1} Q \Psi = (I, 0) U^\top \Phi \quad (29)$$

with the identity matrix $I \in \mathbb{R}^{r \times r}$. Hence we have identified an orthonormal basis $\Psi^* = (\Psi_1^*, \dots, \Psi_r^*)^\top$ satisfying $\text{span}\{\Psi_1^*, \dots, \Psi_r^*\} = \text{span}\{\Psi_1, \dots, \Psi_r\}$. On the one hand, the basis functions Ψ^* can be calculated from the original basis Φ via (29) using the matrix U . On the other hand, it holds that

$$\bar{y}^{(I)}(t, p) = \sum_{j=1}^r \bar{v}_j(t) \Psi_j(p) = \sum_{\ell=1}^r \left[\sum_{j=1}^r s_{\ell} q_{\ell j} \bar{v}_j(t) \right] \Psi_\ell^*(p) =: \sum_{\ell=1}^r \bar{v}_\ell^*(t) \Psi_\ell^*(p) \quad (30)$$

with new coefficient functions \bar{v}_ℓ^* computable from original coefficient functions \bar{v}_j by the matrix $Q = (q_{ij})$ due to $\Psi = Q^\top S' \Psi^*$. Hence an explicit calculation of the basis functions Ψ is never required. Now we define an approximation by a truncation of the finite sum in (30), i.e.,

$$\check{y}^{(r')}(t, p) := \sum_{\ell=1}^{r'} \left[\sum_{j=1}^r s_{\ell} q_{\ell j} \bar{v}_j(t) \right] \Psi_\ell^*(p) \quad (31)$$

for an $r' \in \{1, \dots, r-1\}$. In our context, the following quality of the approximation is guaranteed.

Theorem 3 *The output (30) and its approximation (31) satisfy the error estimates*

$$\left\| \bar{y}^{(I)}(t, \cdot) - \check{y}^{(r')}(t, \cdot) \right\|_{\mathcal{L}^2(\Omega)} \leq \sqrt{r - r'} s_{r'+1} \sqrt{\sum_{j=1}^r \bar{v}_j(t)^2} \quad \text{for } t \geq 0 \quad (32)$$

$$\left\| \bar{y}^{(I)} - \check{y}^{(r')} \right\|_{\mathcal{L}^2(\Omega) \times \mathcal{L}^2[0, \infty)} \leq \sqrt{r - r'} s_{r'+1} \sqrt{\sum_{j=1}^r \|\bar{v}_j\|_{\mathcal{L}^2[0, \infty)}^2} \quad (33)$$

with $r' = 1, \dots, r-1$.

Proof:

We obtain

$$\bar{y}^{(I)}(t, p) - \check{y}^{(r')}(t, p) = \sum_{\ell=r'+1}^r \left[\sum_{j=1}^r s_{\ell} q_{\ell j} \bar{v}_j(t) \right] \Psi_\ell^*(p)$$

for each $t \geq 0$ and $p \in \Pi$. The orthonormality of the basis functions implies

$$\left\| \bar{y}^{(I)}(t, \cdot) - \check{y}^{(r')}(t, \cdot) \right\|_{\mathcal{L}^2(\Omega)}^2 = \sum_{\ell=r'+1}^r \left[\sum_{j=1}^r s_{\ell} q_{\ell j} \bar{v}_j(t) \right]^2.$$

The Cauchy-Schwarz inequality and the orthogonality of the matrix Q yields

$$\left(\sum_{j=1}^r s_\ell q_{\ell j} \bar{v}_j(t) \right)^2 \leq s_\ell^2 \left(\sum_{j=1}^r q_{\ell j}^2 \right) \left(\sum_{j=1}^r \bar{v}_j(t)^2 \right) \leq s_{r'+1}^2 \sum_{j=1}^r \bar{v}_j(t)^2$$

uniformly for $\ell = r' + 1, \dots, r$. Taking the sum of all $\ell = r' + 1, \dots, r$ shows the estimate (32). Employing the integral on the time domain $[0, \infty)$ confirms the estimate (33). \square

The time-dependent part of the upper bounds in (32),(33) cannot be estimated further without additional assumptions. However, we apply this reduction step only to remove a numerical rank deficiency, where r' is selected such that the dominant singular value $s_{r'+1}$ is tiny but still significantly above the machine precision.

Only the first r columns of the matrix $U = (u_{ij})$ are applied to determine the new orthonormal basis in (29). Hence these columns provide a measure for the influence of an original basis function Φ_i within the new basis. We define the values

$$\kappa_i := \sqrt{\sum_{j=1}^r u_{ij}^2} \quad \text{for } i = 1, \dots, m. \quad (34)$$

Since the matrix U is orthogonal, it holds that $0 \leq \kappa_i \leq 1$ for all i and $r \leq m$. This measure of is independent of the quantification by Hardy norms as done in Section 3.

5 Illustrative example

Now we apply the reduction approaches from the previous sections to a test example. All numerical calculations are performed within the software package MATLAB (version R2014b), where the machine precision is around $\varepsilon_0 = 2 \cdot 10^{-16}$.

5.1 Modelling of a low pass filter

As test example, we investigate the electric circuit of a low pass filter, see Figure 1. The electric circuit includes 7 capacitances, 6 inductances and 8 conductances. Thus $q = 21$ physical parameters occur. The system is SISO, because a single input voltage is supplied and the output voltage drops at a load conductor. A mathematical modelling yields a system (1) of DAEs for the 14 node voltages

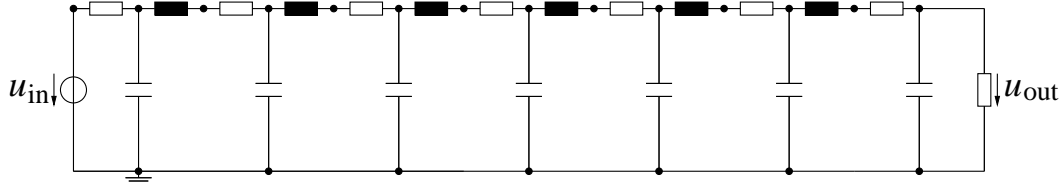


Figure 1: Electric circuit of a low pass filter.

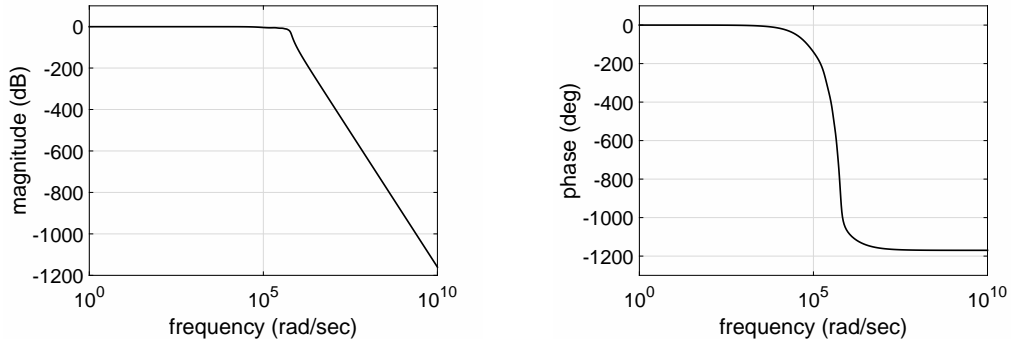


Figure 2: Bode plot of the low pass filter for constant physical parameters: magnitude (left) and phase (right) of the transfer function.

and the 6 currents through the inductances. Hence the dimension of the inner variables is $n = 20$. The (nilpotency) index of the DAE system results to one. Furthermore, the linear dynamical system is stable as well as strictly proper. Figure 2 depicts the bode plot of this linear time-invariant system in the case of a constant choice for the parameters. The magnitude of the transfer function demonstrates the characteristics of a low pass filter.

In the stochastic modelling, we replace all physical parameters by uniformly distributed random variables with a range of 10% around their mean values. The mean values are the constant choice of the parameters from above. In the orthogonal expansion, we include all multivariate polynomials up to total degree $d = 3$, which results in $m = 2024$ basis functions. The stochastic Galerkin method generates a linear dynamical system (8) of dimension $mn = 40480$. We compute the involved matrices numerically, where the probabilistic integrals are approximated by a quadrature on a sparse grid with 13329 nodes in the domain Π . The output matrix \hat{C} is obtained directly, since the matrix C in (1) does not depend on the physical parameters. Moreover, \hat{C} exhibits full (numerical) rank. Numerical computations confirm that the resulting linear dynamical system (8) is stable as well as strictly proper.

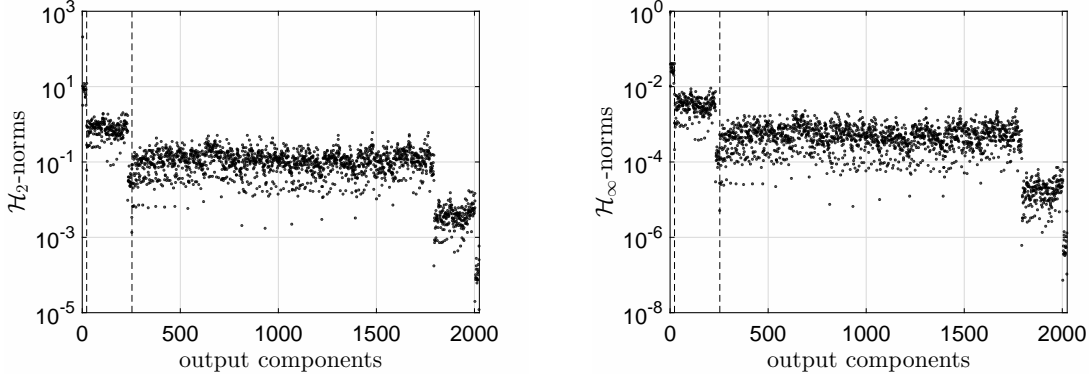


Figure 3: \mathcal{H}_2 -norms (left) and \mathcal{H}_∞ -norms (right) for the components of the transfer function in the stochastic Galerkin system. (The two dashed lines separate the components for polynomials of degree zero/one, degree two and degree three.)

5.2 Sparsification of the given basis

We compute the \mathcal{H}_2 -norms as well as \mathcal{H}_∞ -norms for the separate components of the transfer function associated to the stochastic Galerkin system (8). For this purpose, the transfer function is evaluated on a logarithmically spaced grid $0 \leq \omega_1 < \omega_2 < \dots < \omega_k$ inside the imaginary axis $\{i\omega : \omega \in \mathbb{R}\}$ due to symmetry reasons. For the \mathcal{H}_∞ -norm (13), just the maximum of the finite set of absolute values is chosen as approximation. For the \mathcal{H}_2 -norm (15), the trapezoidal rule yields an approximation to the integral. In each frequency point $s = i\omega$, the computational effort for an evaluation of (9) is dominated by the solution of a linear system of algebraic equations with coefficient matrix $s\hat{E} - \hat{A}$ and right-hand side \hat{B} . Although the number of outputs is large, the computational work of the matrix-vector-multiplication with the sparse matrix \hat{C} is negligible. The resulting norms are depicted for all output components in Figure 3. Furthermore, Figure 4 (left) shows these norms in a descending sequence. We observe different orders of magnitudes for the norms, which mainly depend on the degree of the associated basis polynomials. Nevertheless, there are several components for degree two as well as degree three, whose norms have the same magnitude.

These results indicate some potential for a sparse approximation (18) as explained in Section 3.1. On the one hand, the upper bounds from Theorem 1 could be evaluated for some selections of index sets. We omit the presentation for shortness. On the other hand, Figure 5 illustrates the ratios θ_r from (21) for index sets \mathcal{I}'_r with $r = 1, \dots, 200$. We recognise that the values θ_r tend to one. Furthermore, Table 1 shows the minimum cardinality $r = |\mathcal{I}'_r|$ and the accompanying ratio r/m , which are necessary to achieve a value $\theta_r \geq 1 - \delta$ for several thresholds $\delta > 0$.

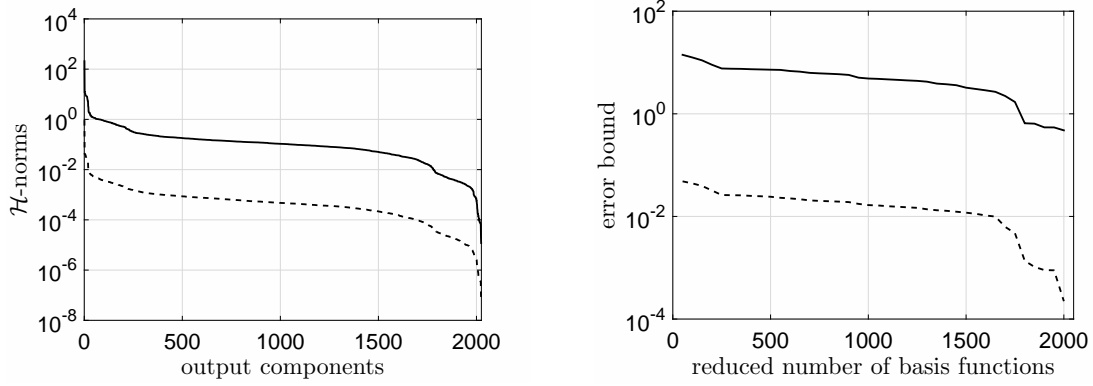


Figure 4: Hardy norms of the stochastic Galerkin system in descending order (left) and error bounds (23),(24) for reduced Galerkin system (right). Solid lines and dashed lines show data for \mathcal{H}_2 -norms and \mathcal{H}_∞ -norms, respectively.

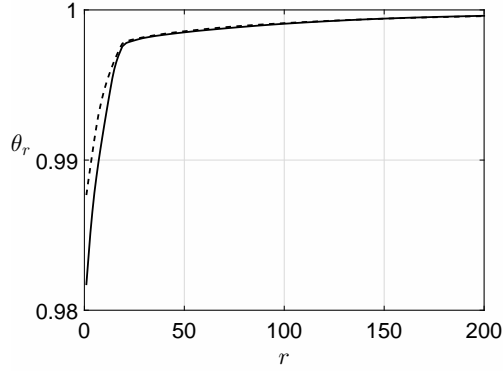


Figure 5: Ratios θ_r from (21) for different index sets \mathcal{I}_r' . Solid lines and dashed lines show data for \mathcal{H}_2 -norms and \mathcal{H}_∞ -norms, respectively.

Table 1: Minimum cardinality r required for a value $\theta_r \geq 1 - \delta$ in (21) and their ratios r/m in percentage.

		$\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$	$\delta = 10^{-5}$
\mathcal{H}_2 -norms	r	8	92	672	1350
	r/m	0.4%	4.6%	33.2%	66.7%
\mathcal{H}_∞ -norms	r	4	86	678	1331
	r/m	0.2%	4.3%	33.5%	65.8%

Table 2: Minimum dimensions required for an error bound below a threshold δ .

	$\delta = 10^{-1}$	$\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$
bound (23)	53	57	67	80
bound (24)	27	45	53	56

We also use the reduction technique from Section 3.2, where the stochastic Galerkin system is shortened. In Table 1, the ratios also indicate the reduction of the dimensionality for the Galerkin system. Furthermore, we perform this reduction for a sequence of index sets \mathcal{I}'_r with $10 \leq r \leq 2000$. The upper error bounds (23),(24) from Theorem 2 are evaluated assuming an input with unit norm, where the Hardy norms are approximated by the same approach as above. Figure 4 (right) depicts the results. In agreement to Figure 4 (left), we observe that the error estimates are bounded from below by (25). Hence only a moderate potential for a sparse approximation and a reduction of the stochastic Galerkin system occurs.

5.3 New basis from MOR

Now the approach from Section 4.1 is employed. We use a moment matching technique with a single expansion point in the frequency domain, see [11, Sect. 3.4]. A Krylov subspace is defined by the input vector \hat{B} . In our MOR technique, the output matrix \hat{C} is not involved and thus the computational effort is independent of the number of outputs. The Arnoldi algorithm yields a projection matrix T_r , whose columns form an orthonormal basis. We simply choose $T_1 = T_r$. In the system (10), the smaller matrices follow from (11). We tried several choices for the expansion point on the real axis. The best instance resulted to $s = 5 \cdot 10^5$. We can select an arbitrary dimension $r \leq nm$ of the reduced system (10) by taking the first r vectors of the Arnoldi process. Numerical computations confirm that the reduced systems (10) are stable for $r = 52, \dots, 100$ and instable for some $r \leq 51$. Moreover, numerical examinations show that the systems (10) are strictly proper in all observed cases.

Figure 6 illustrates the upper error bounds (23),(24) for a unit norm of the input in the case of dimensions $r = 10, \dots, 100$. The minimum dimensions, which are required to achieve error estimates below some thresholds, are given in Table 2. We observe a rapid decay of the error bounds, which confirms a high potential for a reduction of the dimensionality in the linear dynamical system (10). Other MOR techniques, which involve the output matrix \hat{C} , most likely generate even lower-dimensional systems (10) of the same accuracy.

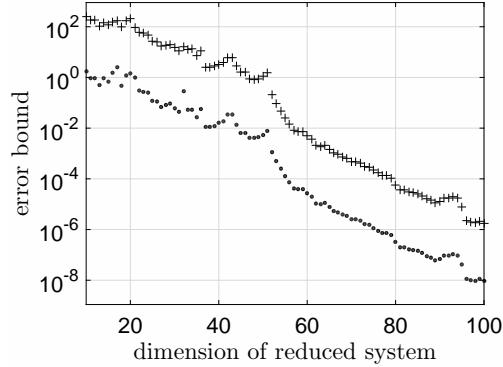


Figure 6: Error bounds from (23) with \mathcal{H}_2 -norms (+) and (24) with \mathcal{H}_∞ -norms (\cdot) for different dimensions of the reduced system.

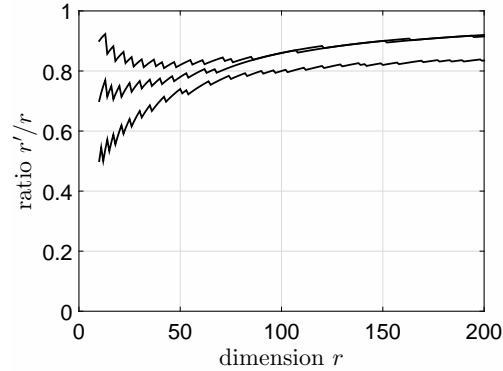


Figure 7: Ratios between the dimension r of the reduced systems and the dimension r' after a further reduction by the omission of all singular values below the thresholds 10^{-4} (lower line), 10^{-8} (center line) and 10^{-12} (upper line).

We also use the technique described in Section 4.2, where the SVD of the output matrix is computed. An additional reduction from the number r in (30) to a lower number r' motivated by Theorem 3 is feasible by neglecting all singular values below some threshold. Figure 7 shows the ratios r'/r for different dimensions r and three choices of the threshold. We recognise some gain in efficiency by removing the unessential parts. For example, a dimension $r = 100$ can be decreased further to $r' = 80$ -90 depending on the required accuracy.

Finally, we compute the values κ_i from (34) using the SVD (28), where the reduced system exhibits the dimension $r = 50$. The results are depicted in Figure 8. We recover the structure of the Hardy norms for the original stochastic Galerkin system (8), cf. Figure 3, although the computation of the values (34) is done completely different. Yet some components for basis functions of degree two

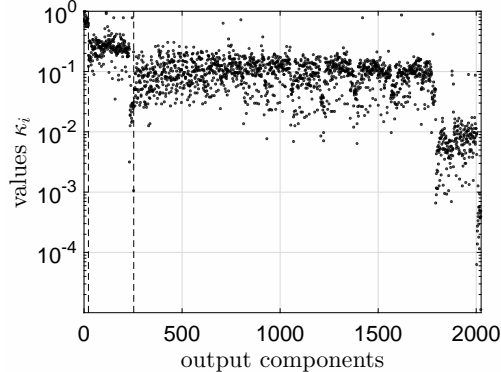


Figure 8: Values κ_i from (34) for all output components in the case of a reduced system with dimension $r = 50$. (The two dashed lines separate the components for polynomials of degree zero/one, degree two and degree three.)

and three feature relatively high numbers (close to the upper bound one) now. This property indicates that some components, which are less important in the system (8), become crucial for achieving our sparse approximation.

6 Conclusions and outlook

We examined two numerical techniques for the identification of a sparse representation, which approximates a quantity of interest from a random linear dynamical system. On the one hand, terms of an orthogonal expansion were omitted if their accompanying Hardy norms are relatively small. On the other hand, a projection-based model order reduction yields an alternative orthogonal expansion with a low number of basis functions. We proved upper error bounds for the sparse approximations in both techniques. In addition, we carried out numerical computations for a test example with high dimensionality. The results indicate that the second approach is more efficient than the first method. If the same accuracy is required with respect to the error bounds, then the second technique yields a sparse approximation with a lower number of basis functions.

A topic for further research is the construction of a sparse approximation in the case of random nonlinear dynamical systems. Model order reduction becomes harder in the nonlinear situation. For example, a strict estimation of approximation errors is critical or even impossible. As a simplification, one could investigate a linear dynamical system, whose quantity of interest depends nonlinearly on the state variables or the inner variables, respectively.

References

- [1] A. Antoulas, Approximation of Large-Scale Dynamical Systems, SIAM Publications, 2005.
- [2] F. Augustin, A. Gilg, M. Paffrath, P. Rentrop, U. Wever, Polynomial chaos for the approximation of uncertainties: chances and limits, Euro. Jnl. of Applied Mathematics 19 (2008) 149–190.
- [3] P. Benner, M. Hinze, E.J.W. ter Maten (eds.), Model Reduction for Circuit Simulation, Lect. Notes in Electr. Engng. Vol. 74, Springer, 2011.
- [4] P. Benner, S. Gugercin, K. Willcox, A survey of projection-based model order reduction methods for parametric dynamical systems, SIAM Review 57 (2015) 483–531.
- [5] G. Blatman, B. Sudret, Adaptive sparse polynomial chaos expansion based on least angle regression, J. Comput. Phys. 230 (2011) 2345–2367.
- [6] P.R. Conrad, Y.M. Marzouk, Adaptive Smolyak pseudospectral approximations, SIAM J. Sci. Comput. 35 (2013) A2643–A2670.
- [7] A. Doostan, H. Owhadi, A non-adapted sparse approximation of PDEs with stochastic inputs, J. Comput. Phys. 230 (2011) 3015–3034.
- [8] J. Doyle, B. Francis, A. Tannenbaum, Feedback Control Theory, Macmillan, New York, 1992.
- [9] O.G. Ernst, A. Mugler, H.J. Starkloff, E. Ullmann, On the convergence of generalized polynomial chaos expansions, ESAIM: Mathematical Modelling and Numerical Analysis 46 (2012) 317–339.
- [10] L. Feng, Parameter independent model order reduction, Math. Comput. Simulat. 68 (2005) 221–234.
- [11] R. Freund, Model reduction methods based on Krylov subspaces, Acta Numerica 12 (2003) 267–319.
- [12] G.H. Golub, C.F. van Loan, Matrix Computations (3rd ed.), Johns Hopkins University Press, 1996.
- [13] S. Gugercin, A.C. Antoulas, C. Beattie, \mathcal{H}_2 model reduction for large-scale linear dynamical systems. SIAM J. Matrix Anal. Appl. 30 (2008) 609–638.
- [14] S. Gugercin, T. Stykel, S. Wyatt, Model reduction of descriptor systems by interpolatory projection methods. SIAM J. Sci. Comput. 35 (2013) B1010–B1033.

- [15] J.D. Jakeman, M.S. Eldred, K. Sargsyan, Enhancing ℓ_1 -minimization estimates of polynomial chaos expansions using basis selection, *J. Comput. Phys.* 289 (2015) 18–34.
- [16] J.D. Jakeman, A. Narayan, T. Zhou, A generalized sampling and preconditioning scheme for sparse approximation of polynomial chaos expansions, *arXiv: 1602.06879v1* (2016).
- [17] N. Mi, S.X.-D. Tan, P. Liu, J. Cui, Y. Cai, X. Hong, Stochastic extended Krylov subspace method for variational analysis of on-chip power grid networks, in: *Proc. ICCAD 2007*, pp. 48–53.
- [18] P.B. Nair, A.J. Keane, Stochastic reduced basis methods, *AIAA J.* 40 (2002) 1653–1664.
- [19] R. Pulch, Polynomial chaos for linear differential algebraic equations with random parameters, *Int. J. Uncertain. Quantif.* 1 (2011) 223–240.
- [20] R. Pulch, Stochastic collocation and stochastic Galerkin methods for linear differential algebraic equations, *J. Comput. Appl. Math.* 262 (2014) 281–291.
- [21] R. Pulch, E.J.W. ter Maten, F. Augustin, Sensitivity analysis and model order reduction for random linear dynamical systems, *Math. Comput. Simulat.* 111 (2015) 80–95.
- [22] R. Pulch, E.J.W. ter Maten, Stochastic Galerkin methods and model order reduction for linear dynamical systems, *Int. J. Uncertain. Quantif.* 5 (2015) 255–273.
- [23] R. Pulch, Model order reduction for stochastic expansions of electric circuits, in: A. Bartel, M. Clemens, M. Günther, E.J.W. ter Maten (eds.), *Scientific Computing in Electrical Engineering SCEE 2014. Mathematics in Industry Vol. 23*, Springer, 2016, pp. 223–232.
- [24] S.K. Sachdeva, P.B. Nair, A.J. Keane, Hybridization of stochastic reduced basis methods with polynomial chaos expansions, *Probabilistic Eng. Mech.* 21 (2006) 182–192.
- [25] W.H.A. Schilders, M.A. van der Vorst, J. Rommes (eds.), *Model Order Reduction: Theory, Research Aspects and Applications*, *Mathematics in Industry Vol. 13*, Springer, 2008.
- [26] B. Sonday, R. Berry, B. Debusschere, H. Najm, Eigenvalues of the Jacobian of a Galerkin-projected uncertain ODE system, *SIAM J. Sci. Comput.* 33 (2011) 1212–1233.

- [27] D. Xiu, G.E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations, *SIAM J. Sci. Comput.* 24 (2002) 619–644.
- [28] D. Xiu, J.S. Hesthaven, High order collocation methods for differential equations with random inputs, *SIAM J. Sci. Comput.* 27 (2005) 1118–1139.
- [29] D. Xiu, Numerical methods for stochastic computations: a spectral method approach, Princeton University Press, 2010.
- [30] Y. Zou, Y. Cai, Q. Zhou, X. Hong, S.X.-D. Tan, L. Kang, Practical implementation of the stochastic parameterized model order reduction via Hermite polynomial chaos, in: *Proc. ASP-DAC 2007*, pp. 367–372.